Calculating Heegaard-Floer Homology by Counting Lattice Points in Tetrahedra

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Abstract

We introduce a notion of complexity for Sefiert homology spheres by establishing a correspondence between lattice point counting in tethrahedra and the Heegaard-Floer homology. This complexity turns out to be equivalent to a version of Casson invariant and it is monotone under a natural partial order in the set of Seifert homology spheres. Using this interpretation we prove that there are finitely many Seifert homology spheres with prescribed Heegaard-Floer homology. As an application, we characterize L-spaces and weakly elliptic manifolds among Seifert homology spheres. Also, we list all the Seifert homology spheres up to complexity two.

1 Introduction

Heegaard-Floer homology, introduced by Ozsváth and Szabó in [14] and [13] is a prominent invariant for 3-manifolds. The goal of our article is to explore Heegaard-Floer homology from a combinatorial point of view in the special case of Seifert fibered homology spheres. Although it is more geometric than similar theories, such as Donaldson, or Seiberg-Witten theories, the definition of Heegaard-Floer homology involves a count of a certain moduli space of holomorphic disks into a symmetric product of a surface, which is in general a challenging analytical problem. On the other hand for a certain class of manifolds, namely plumbed manifolds with at most one bad

vertex, works of Ozsváth and Szabó [12], and Nemethi [8] show that the calculation of Heegaard-Floer homology is a purely combinatorial problem. This class of manifolds is relatively small, but it is still large enough to include all Seifert fibered spaces (over S^2).

In [8], for a fixed plumbed 3-manifold, Nemethi finds an explicit algorithm whose output determines the Heegaard-Floer homology completely. See [12] for an alternative algorithm. However, computing Heegaard-Floer homology for infinite families of 3-manifolds seems to be a formidable combinatorial problem for one has to determine all the local maxima and minima of infinite families of sequences which simultaneously solve an infinite family of non-homogeneous recurrence relations. See [9], [7], and [16] for some particular cases where this problem is handled.

To elaborate on the problem mentioned in the previous paragraph, let us briefly review Nemethi's method in the simplest case, where the 3-manifold is a Seifert homology sphere. Let $p_1, \ldots, p_l$ be a list of pairwise relatively prime integers such that $1 < p_1 < p_2 < \cdots < p_l$. We denote by $\Sigma(p_1, p_2, \ldots, p_l)$ the Seifert fibered 3-manifold over S^2 with l singular fibers whose Seifert invariants are given by $(e_0, (p_, p_1'), (p_2, p_2'), \ldots, (p_l, p_l'))$. Here, $(x_0, x_1, \ldots, x_l) = (e_0, p_1', p_2', \ldots, p_l')$ is the unique solution to the Diophantine equation

$$x_0 p_1 p_2 \cdots p_l + x_1 p_2 \cdots p_l + p_1 x_2 \cdots p_l + \cdots + p_1 p_2 \cdots x_l = -1, \tag{1.1}$$

where $1 \le x_i \le p_i - 1$, for $i = 1, 2, \ldots, l$. (1.1) guarantees that $\Sigma(p_1, p_2, \ldots, p_l)$ has trivial first homology, so it is an integral homology sphere.

To calculate Heegaard-Floer homology of $\Sigma(p_1, p_2, \ldots, p_l)$, we consider the sequence $\tau : \mathbb{N} \to \mathbb{Z}$ defined by the recurrence

$$\tau(n+1) = \tau(n) + 1 + |e_0|n - \sum_{i=1}^{l} \left\lceil \frac{np_i'}{p_i} \right\rceil \tag{1.2}$$

with the given initial condition $\tau(0) = 0$. Here $\lceil y \rceil$ represents the minimum integer larger than y. We say that $\tau(n_0)$ is a *local maximum* of τ, if there exist integers a, b such that $a < n_0 < b$ with $\tau(a) < \tau(n_0) > \tau(b)$, and τ is monotone increasing on the interval $[a, n_0]$ and monotone decreasing on $[n_0, b]$. *Local minimum* values of τ are defined similarly. It turns out that, up to a degree shift, the Heegaard-Floer homology is determined by the subsequence τ' of τ consisting of all local minima and local maxima .

In our first result we analyze the difference term in (1.2) in order to understand the local extrema of τ . For notational convenience we focus our attention to Brieskorn spheres, which are by definition the Seifert homology spheres with three singular fibers. Nevertheless, most of our arguments are adaptable for studying arbitrary number of singular fibers with some notational changes.

Theorem 1.3. *Let (p, q, r) be a triple of pairwise relatively prime integers with $1 < p < q < r$. Define $\Delta : \mathbb{N} \to \mathbb{Z}$*

$$\Delta(n) = 1 + |e_0|n - \left\lceil \frac{np'}{p} \right\rceil - \left\lceil \frac{nq'}{q} \right\rceil - \left\lceil \frac{nr'}{r} \right\rceil,$$

where (e_0, p', q', r') is defined by

$$e_0 pqr + p'qr + pq'r + pqr' = -1$$

with $0 \le p' \le p - 1$, $0 \le q' \le q - 1$, $0 \le r' \le r - 1$. Define the constant

$$N_0 = pqr - pq - qr - pr.$$

Suppose $(p, q, r) \neq (2, 3, 5)$. Then the following holds.

1. *N_0 is a positive integer.*
2. *$\Delta(n) \ge 0$, for all $n > N_0$.*
3. *$\Delta(n) = -\Delta(N_0 - n)$, for all n with $0 \le n \le N_0$.*
4. *$\Delta(n) \in \{-1, 0, 1\}$, for all n with $0 \le n \le N_0$.*
5. *For $0 \le n \le N_0$, one has $\Delta(n) = 1$ if and only if either $n = 0$, or n is an element of the numerical semigroup $G(pq, pr, qr)$ minimally generated by pq, qr, and pr.*

If $(p, q, r) = (2, 3, 5)$, then $\Delta(n) \ge 0$ for all $n \in \mathbb{N}$.

This theorem provides us with a fast and practical means for the calculation of Heegaard-Floer homology of Brieskorn spheres. More importantly it gives a partial answer to the realization problem which we explain in the sequel.

Recall that the Heegaard-Floer homology $HF^+(Y)$ of any closed oriented 3-manifold Y splits as a direct sum over Spin^{c}structures of Y. Each summand admits the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}[U]$-module, where U is a formal variable. If $\mathfrak{s}$ is a Spin^{c}structure with torsion first Chern class then the $\mathbb{Z}/2\mathbb{Z}$-grading on the corresponding summand $HF^+(Y, \mathfrak{s})$ lifts to an absolute $\mathbb{Q}$-grading such that U has degree -2. Integral homology spheres admit unique Spin^{c}structure which is necessarily torsion, and the corresponding absolute $\mathbb{Q}$-grading is in fact a $\mathbb{Z}$-grading.

Question 1.4. Which $\mathbb{Z}$-graded $\mathbb{Z}[U]$-modules can be realized as the Heegaard-Floer homology of a Seifert homology sphere?

It is known that Heegaard-Floer homology of any integral homology sphere splits into $Z[U]$-submodules in the following way: $HF^+(Y) = \mathcal{T}^+_{(d)} \oplus HF_{\mathrm{red}}(Y)$. Here, $\mathcal{T}^+_{(d)}$ is a copy of $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$, on which we impose a grading so that the minimal degree is d. Furthermore, $HF_{\mathrm{red}}(Y)$ is finitely generated. If Y is a Seifert homology sphere oriented so that it bounds a positive definite plumbing, then $HF^+(Y)$ is supported at even degrees.

Let us illustrate how Theorem 1.3 is useful in the processes of solving Question 1.4. Recall that a 3-manifold Y is said to be an *L-space*, if it is a rational homology sphere, and $HF_{\mathrm{red}}(Y, \mathfrak{s}) = 0$ for every Spin^{c}structure $\mathfrak{s}$, [11].

Conjecture 1.5. If an irreducible integral homology sphere Y is an L-space, then Y is the 3-sphere, or the Poincaré homology sphere $\Sigma(2, 3, 5)$ with either orientation.

This conjecture is verified for Seifert homology spheres independently by Rustamov [15] and Eftekhary [4]. There is also an implicit proof of the same statement when one combines the results of [3] with [6]. Here we give an alternative, elementary proof.

Theorem 1.6. *([15], [4]) If a Seifert homology sphere Y is an L–space, then Y is homeomorphic to S^3 or $\pm\Sigma(2, 3, 5)$.*

Among Seifert manifolds, L-spaces are precisely those 3-manifolds with $\Delta(n) \geq 0$ for all $n \geq \mathbb{N}$. Therefore, Theorem 1.3 is sufficient to prove Theorem 1.6 in the case of three singular fibers. The case of arbitrarily many singular fibers follows from an extension of our theorem to that setting.

Above results suggest that the sum of negative values of the Δ function is a significant quantity for it defines a kind of “complexity” for the Heegaard-Floer homology. Indeed, what we observe above is that the complexity 0 Seifert manifolds are precisely the L-spaces. Therefore, our next definition is meaningful.

Definition 1.7. Let $p_1, \ldots, p_l$ be a list of relatively prime integers such that $1 < p_1 < \cdots < p_l$. For $\tau(n)$ as defined in (1.2), put $\Delta(n) = \tau(n + 1) - \tau(n)$. We define

$$\kappa(p_1, p_2, \ldots, p_l) = \left| \sum_{i=0}^{\infty} \min\{0, \Delta(n)\} \right|.$$

It follows from [8] that if two Heegaard-Floer homology groups $HF^+(\Sigma(p_1, p_2, \ldots, p_l))$ and $HF^+(\Sigma(q_1, q_2, \ldots, q_l))$ are isomorphic, then the corresponding *kappa* invariants $\kappa(p_1, p_2, \ldots, p_l)$ and $\kappa(q_1, q_2, \ldots, q_l)$ are equal. The converse does not hold in general, however, as we show, there are only finitely many isomorphism types of $\mathbb{Z}[U]$-modules which can appear as the Heegaard-Floer homology of a Seifert homology sphere with a prescribed κ.

Theorem 1.8. *For any positive integer k, there exists finitely many tuples $(p_1, p_2, \ldots, p_l)$ such that $\kappa(p_1, p_2, \ldots, p_l) = k$, where $p_1, p_2, \ldots, p_l$ are pairwise relatively prime, and $1 < p_1 < p_2 < \cdots < p_l$. Consequently, there exist at most finitely many Seifert homology spheres with prescribed Heegaard-Floer homology.*

By contrast, we should mention that it is possible to find many infinite families of irreducible integral homology spheres with isomorphic Heegaard-Floer homology. See, for example, Proposition 1.2 of [1].

The most important ingredient in the proof of Theorem 1.8 is the monotonicity property of κ with respect to a partial ordering on the set of tuples. We prove this property in propositions 3.10, 4.2, and 4.6. These results together with sufficient computational power, allows one to list all the graded $\mathbb{Z}[U]$-modules that could appear as the Heegaard-Floer homology of a Seifert homology sphere up to a given complexity. In the following theorem, we give this list up to $\kappa = 2$.

Theorem 1.9. *Table 1 contains the list of all graded $\mathbb{Z}[U]$-modules that are isomorphic to a Heegaard-Floer homology for some Seifert homology sphere with $\kappa \leq 2$. Additionally, for each such $\mathbb{Z}[U]$-module M, the table contains the list of all Seifert homology spheres whose Heegaard-Floer homology is M.*

Brieskorn Sphere Y	κ	$d(-Y)$	$HF^+(-Y)$
S^3	0	0	$\mathcal{T}^+_{(0)}$
$\Sigma(2,3,5)$	0	-2	$\mathcal{T}^+_{(-2)}$
$\Sigma(2,3,7)$	1	0	$\mathcal{T}^+_{(0)} \oplus \mathbb{Z}_{(0)}$
$\Sigma(2,3,11)$	1	-2	$\mathcal{T}^+_{(-2)} \oplus \mathbb{Z}_{(-2)}$
$\Sigma(2,3,13)$, $\Sigma(2,5,7)$, $\Sigma(3,4,5)$	2	0	$\mathcal{T}^+_{(0)} \oplus \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(0)}$
$\Sigma(2,3,17)$, $\Sigma(2,5,9)$	2	-2	$\mathcal{T}^+_{(-2)} \oplus \mathbb{Z}_{(-2)} \oplus \mathbb{Z}_{(-2)}$

Table 1: Seifert homology spheres with $\kappa \leq 2$.

Note that Theorem 1.6 is a special case of Theorem 1.9 for which $\kappa = 0$.

Using a relation between the Euler characteristic of Heegaard-Floer homology and the Casson invariant, we relate κ to other well known invariants of Seifert homology spheres. Recall that every Seifert fibered space is the boundary of a 4-manifold that is plumbing of disk bundles over spheres, where the plumbing is done according to a negative definite star shaped weighted tree.

Such a plumbing configuration is unique up to blow-up and blow-down. Suppose we fix one such plumbing and let s denote the number of its vertices. Also, let K denote its canonical cohomology class. Then the number $K^2 + s$ is invariant under blow-up and blow-down, so it defines an invariant for the Seifert fibered space.

Proposition 1.10. *For a Seifert homology sphere $Y = \Sigma(p_1, p_2, \ldots, p_l)$, the following equality holds*

$$\kappa(p_1, p_2, \ldots, p_l) = \lambda(-Y) - (K^2 + s)/8,$$

where $\lambda(-Y)$ is the Casson invariant of $-Y$ [2], normalized so that the Poincaré homology sphere $\Sigma(2, 3, 5)$ oriented as the boundary of negative E_8 plumbing satisfies $\lambda(-\Sigma(2, 3, 5)) = -1$.

In our next result we observe a remarkable connection between κ, numerical semigroups and lattice points in tetrahedra. Casson invariants of Brieskorn spheres have similar interpretations. See [5]. We state it for the special case of 3-singular fibers here. There is also a more technical statement that works for arbitrary number of singular fibers which we state in Theorem 4.3.

Theorem 1.11. *Given a Brieskorn sphere $\Sigma(p, q, r)$ with defining integers $1 < p < q < r$, $\kappa(p, q, r)$ is equal to the number of lattice points inside the tetrahedron with vertices $(0, 0, 0)$, $(N_0/pq, 0, 0)$, $(0, N_0/pr, 0)$, and $(0, 0, N_0/qr)$, where $N_0 = pqr - pq - pr - qr$. In other words, $\kappa(p, q, r)$ equals the cardinality of the set $G \cap [0, N_0]$, where $G = G(0, pq, pr, qr)$ is the numerical semigroup generated by 0, pq, qr, and pr.*

We push our techniques further to study a class of Brieskorn spheres that has a simple Heegaard-Floer homology. The following definition is due to Nemethi [8].

Definition 1.12. A rational homology sphere Y which is the boundary of a negative definite plumbing tree with at most one bad vertex is said to be *weakly elliptic*, if its Heegaard-Floer homology in the canonical Spin^{c}structure is of the form $T^+_{(d)} \oplus (\mathbb{Z}_{(d)})^l$ for some $l \geq 1$ and some even integer d.

It is shown in Proposition 6.5 of [8] that, if Y weakly elliptic, then it is the link of a weakly elliptic singularity. Our next result gives the complete list of weakly elliptic Brieskorn spheres.

Theorem 1.13. *A Brieskorn sphere $\Sigma = \Sigma(p, q, r)$ is weakly elliptic if and only if (p, q, r) is equal to one of the following triplets; $(3, 4, 5), (2, 5, 7), (2, 5, 9)$, or $(2, 3, r)$ with gcd$(6, r) = 1$ and $r > 5$.*

To further analyze the relationship between lattice point counting and τ of $\Sigma(p_1, \ldots, p_l)$, we compare their generating functions. Indeed our computations show that the generating function of

$\tau(n)$ is a rational function, similar to the generating function of the sequence counting the lattice points on the hyperplane $p_1x_1 + p_2x_2 + \cdots + p_lx_l = n$ that lie in the first orthant of $\mathbb{R}^l$. For the sake of space, here we write only the simplified version of the generating function of $\tau(n)$. See Theorem 7.2 for its explicit form.

Theorem 1.14. *The generating function* $F(x) = \sum_n \tau(n)x^n$ *is given by*

$$F(x) = \frac{G(x)}{(1 - x^{p_1})(1 - x^{p_2}) \cdots (1 - x^{p_l})},$$

where $G(x)$ *is a polynomial in* x *with degree less than or equal to* $p_1+p_2+\cdots+p_l-1$*. Furthermore, if* $|e_0| \neq 1$*, then the degree of* $G(x)$ *is exactly* $p_1 + p_2 + \cdots + p_l - 1$*.*

The organization of our paper is as follows. In Section 2 we review Nemethi's method and see how Heegaard-Floer homology is calculated from the τ-function. We analyze the Δ-function in Section 3, and prove therein Theorem 1.3 and Theorem 1.11. We extend our results to arbitrary number of singular fibers in Section 4. Theorems 1.6, 1.8, and 1.9 are proved in Section 5. In Section 6, we characterize weakly elliptic Brieskorn spheres interms of certain numerical semigroups and prove Theorem 1.13. Finally, we conclude our paper, by calculating the generating function of $\tau(n)$ in Section 7.

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2 Graded Roots and Heegaard-Floer Homology

Here we review the definition of a "graded root," and discuss its basic properties. For more information and background, we recommend [8] and Section 2 of [1].

Definition 2.1. A *graded root* is a pair (R, χ), where R is an infinite tree, and χ is an integer valued function defined on the vertex set $V = V(R)$ of R satisfying the following properties.

1. $\chi(u) - \chi(v) = \pm 1$, if there is an edge connecting u and v.

2. $\chi(u) > \min\{v, w\}$, if there are edges connecting u to v, and u to w.

3. χ is bounded below.

4. $\chi^{-1}(k)$ is finite for every k.

5. $|\chi^{-1}(k)| = 1$ for k large enough.

In Figure 1 we give an example of a graded root, where the infinite tree R is drawn on left, and the function χ is obtained from the heights of the vertices.

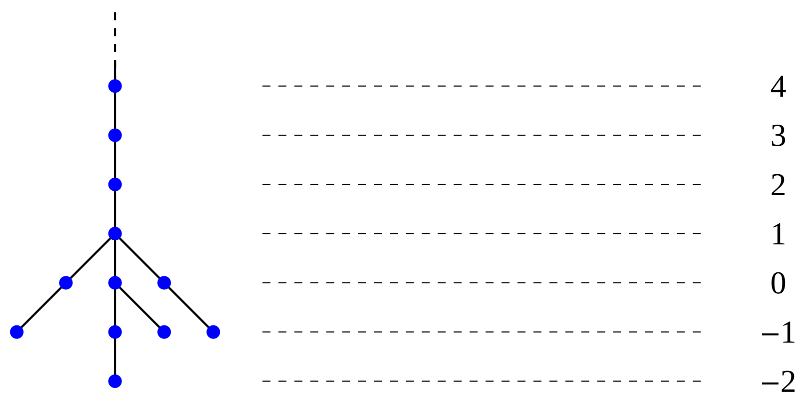

Figure 1: A graded root.

The branches of a graded root are enumerated from left to right, and the vertex at the bottom of the i-th branch is called the *i-th vertex*. The data of a graded root is encoded in a finite sequence $[a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n]$, where a_i is the value of χ of at the i-th vertex, and b_i is the value of χ at the branching vertex connecting i-th branch to the $i+1$-th branch. For example, the graded root given in Figure 1 is represented by the sequence $[-1, 1, -2, 0, -1, 1 - 1]$.

Conversely, any sequence $[a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n]$ satisfying $b_i > \max\{a_i, a_{i+1}\}$, for $i = 1, \ldots, n-1$ determines a graded root. For a given sequence τ with this property, we denote the corresponding graded root by (R_τ, χ_τ).

The nomenclature of "graded root" is explained by the natural correspondence between graded roots and graded $\mathbb{Z}[U]$-modules. Consider the free $\mathbb{Z}$-module generated by the vertex set V. The U action is described as follows. For a given vertex v, $U \cdot v$ has a summand supported at the vertex w, if there is an edge connecting v to w, and $\chi(v) > \chi(w)$. Then the action of U is extended by linearity. Finally, the grading is determined by the requirement that every vertex v has degree $2\chi(v)$. Given a

graded root (R, χ), we denote the associated $\mathbb{Z}[U]$-module by $\mathbb{H}(R, \chi)$. For example, for the graded root given in Figure 1, the associated $\mathbb{Z}[U]$-module is isomorphic to $\mathcal{T}^+_{(-4)} \oplus \mathcal{T}^1_{(-2)} \oplus \mathcal{T}^1_{(-2)} \oplus \mathbb{Z}_{(-2)}$. Here, we use the following notation: $\mathcal{T}^+_{(d)} = \mathbb{Z}[U, U^{-1}]/\langle U\rangle$, and $\mathcal{T}^n_{(d)} = \mathbb{Z}[U]/\langle U^{n+1}\rangle$; both groups are graded so that U has degree -2, and the minimal degree is d.

Fix a Seifert homology sphere $\Sigma(p_1, \ldots, p_l)$, and let τ denote the sequence defined recursively as in (1.2). It is known that $\tau(n)$ is an increasing function of n, for all sufficiently large $n >> 0$. It follows that the subsequence consisting of local minima and local maxima of τ is a finite sequence. By abuse of notation, we denote this finite subsequence by τ, also. Now consider the graded root given by τ and its $\mathbb{Z}[U]$-module $\mathbb{H}(R_\tau, \chi_\tau)$. It turns out that, up to a global degree shift the Heegaard-Floer homology of $\Sigma(p_1, \ldots, p_l)$ is isomorphic to $\mathbb{H}(R_\tau, \chi_\tau)$.

The degree shift is calculated as follows. Let X denote the 4-manifold X bounding $\Sigma(p_1, \ldots, p_l)$, which is a star shaped plumbing of certain disk bundles over 2-sphere with a negative definite intersection form. The second homology of X has a natural basis $e_0, e_1, \ldots, e_{s-1}$ consisting of base spheres. Here, e_0 corresponds to the central vertex in the plumbing graph, and s is equal to the total number of vertices. The canonical 2-cohomology class K is defined by the requirement that $K(e_i) = -e_i \cdot e_i - 2$. Then, the desired degree shift is given by $-(K^2 + s)/4$.

An alternative approach utilizes the "Dedekind sums" for computing the degree shift. The *Dedekind sum*, $s(p, q)$ is calculated recursively by setting $s(1, 1) = 0$ and repeatedly applying the reciprocity law

$$s(p, q) + s(q, p) = -\frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right),$$

and using the rule stating that whenever $r \equiv p \mod q$, the equality $s(p, q) = s(r, q)$ holds.

It is shown in [10] that

$$K^2 + s = \epsilon^2 e + e + 5 - 12 \sum_{i=1}^{l} s(p'_i, p_i), \tag{2.2}$$

where

$$e = e_0 + \sum_{i=1}^{l} \frac{p'_i}{p_i}, \quad \text{and} \quad \epsilon = \left(2 - l + \sum_{i=1}^{l} \frac{1}{p_i}\right)\frac{1}{e}.$$

In conclusion we have the following result.

Theorem 2.3. *([8]) For any Seifert homology sphere $Y := \Sigma(p_1, p_2, \ldots, p_l)$, the Heegaard-Floer homology group $HF^+(-Y)$ is isomorphic to $\mathbb{H}(R_\tau, \chi_\tau)$ with a degree shift $-(K^2 + s)/4$, where τ is the sequence defined in (1.2), and (R_τ, χ_τ) is the associated graded root.*

Example 2.4. Combining Theorem 2.3 and our Theorem 1.3, it is now easy to calculate the Heegaard-Floer homology of a Seifert homology sphere. Let us illustrate this statement on $Y = \Sigma(2, 3, 11)$.

We have $N_0 = 5$. Consider the semigroup $G' = G \cup \{0\}$, where $G = G(6, 22, 33)$ is the numerical semigroup generated by the integers 6, 22, and 33. The only element of G' that is contained in the interval $[0, N_0]$ is 0. Therefore Theorem 1.3 implies $\Delta(0) = 1$, $\Delta(5) = -1$, and $\Delta(n) \geq 0$ for all $n \neq 0, 5$. Hence $\tau(n) = \sum_{i=0}^{n-1} \Delta(i)$ has two local minimum values (both of which are equal to 0) and one local maximum value (which is equal to 1). Consider the graded root (R_τ, χ_τ) associated to the sequence $\tau = [0, 1, 0]$. We have $\mathbb{H}(R_\tau, \chi_\tau) = \mathcal{T}^+_{(0)} \oplus \mathbb{Z}_{(0)}$.

We need to calculate the degree shift $-(K^2+s)/4$. It follows from (1.1) that the Seifert invariants of $\Sigma(2, 3, 11)$ are given by

$$(e_0, (p', p), (q', q), (r', r)) = (-2, (1, 2), (2, 3), (9, 11)).$$

We calculate the terms appearing in (2.2), and see that $e = -1/66$, $\epsilon = 5$. The Dedekind sums are calculated by repeatedly applying the reciprocity law:

$$\begin{aligned} s(1, 2) &= 0, \\ s(2, 3) &= -1/18, \\ s(9, 11) &= -5/22. \end{aligned}$$

Using these values in (2.2), the degree shift is calculated to be $-(K^2 + s)/4 = -2$. Theorem 1.3 says that $HF^+(-\Sigma(2, 3, 11)) = \mathcal{T}^+_{(-2)} \oplus \mathbb{Z}_{(-2)}$.

3 Analysis of the Delta Function

In order to determine the positions and values of the local extrema of τ function, we study its difference term

$$\Delta(n) = 1 + e_0 n - \left\lceil \frac{np'}{p} \right\rceil - \left\lceil \frac{nq'}{q} \right\rceil - \left\lceil \frac{nr'}{r} \right\rceil.$$

Our first task is to write Δ as a quasi-polynomial. To this end, for $x \in \mathbb{R}$, define

$$f(x) := \lceil x \rceil - x.$$

Lemma 3.1. *Given relatively prime integers a and m, the sequence*

$$g(n) = f(na/m) = \lceil na/m \rceil - na/m$$

is periodic with period m. Moreover, the finite sequence $(mg(0), mg(1), \dots, mg(m-1))$ is the same as the orbit of $m-a$ in the additive group $\mathbb{Z}/m\mathbb{Z}$. Consequently, for every $s \in \{0, \dots m-1\}$ there exists unique n such that $0 \le n \le m-1$ and $f(na/m) = s/m$.

Proof. Writing $n = pm + r$ we see that $g(n) = g(r)$, establishing the periodicity of g. For the second part it suffices to observe that $g(n)$ is the fractional part of $n(m-a)/m$. Indeed,

$$\frac{n(m-a)}{m} - \left\lfloor \frac{n(m-a)}{m} \right\rfloor = -\frac{na}{m} - \left\lfloor -\frac{an}{m} \right\rfloor = -\frac{na}{m} + \left\lceil \frac{an}{m} \right\rceil .$$

□

Now we manipulate Δ as follows

$$\Delta(n) = 1 - e_0 n - \frac{np'}{p} - \frac{nq'}{q} - \frac{nr'}{r} - \left(\left\lceil \frac{np'}{p} \right\rceil - \frac{np'}{p} + \left\lceil \frac{nq'}{q} \right\rceil - \frac{nq'}{q} + \left\lceil \frac{nr'}{r} \right\rceil - \frac{nr'}{r} \right).$$

In other words,

$$\Delta(n) = 1 - \frac{e_0 pqr + p'qr + pq'r + pqr'}{pqr} n - \left(f\left(\frac{np'}{p}\right) + f\left(\frac{nq'}{q}\right) + f\left(\frac{nr'}{r}\right) \right).$$

Using (1.1), we conclude that

$$\Delta(n) = 1 + \frac{1}{pqr} n - \left(f\left(\frac{np'}{p}\right) + f\left(\frac{nq'}{q}\right) + f\left(\frac{nr'}{r}\right) \right). \tag{3.2}$$

The advantage of writing Δ in this form is that the quasi-polynomial structure becomes apparent, which is also suggested by the generating function calculation in Section 7. In fact, it follows from generating function calculations that Δ is the sum of a linear polynomial and a periodic function (which, in turn, can be written as the sum of three periodic functions). This form allows us to do the following critical analysis regarding the values of Δ.

Proposition 3.3. *For $0 \le n < pqr$, we have $-1 \le \Delta(n) \le 1$. Moreover,*

1. $\Delta(n) = -1$ *if and only if* $f\left(\frac{np'}{p}\right) + f\left(\frac{nq'}{q}\right) + f\left(\frac{nr'}{r}\right) \ge 2.$

2. $\Delta(n) = 1$ *if and only if* $f\left(\frac{np'}{p}\right) + f\left(\frac{nq'}{q}\right) + f\left(\frac{nr'}{r}\right) \le 1.$

For $n \geq pqr$, we have $\Delta(n) \geq 0$.

Proof. Clearly, $\Delta(0) = 1$. Suppose $0 < n < pqr$. Let $A(n) = f\left(\frac{np'}{p}\right) + f\left(\frac{nq'}{q}\right) + f\left(\frac{nr'}{r}\right)$. We have $0 < A(n) < 3$, and $0 < n/pqr < 1$. Therefore $\Delta(n) = 1 + n/pqr - A(n)$ satisfies

$$-2 < \Delta(n) < 2.$$

Note that $\Delta(n)$ is integer valued, hence $-1 \leq \Delta(n) \leq 1$ for all $n < pqr$. If $A(n) \geq 2$ then $\Delta(n) < 0$, so item 1 follows again from the fact that Δ is integer valued. Similarly, if $A(n) \leq 1$ then $\Delta > 0$, so we obtain the second item.

For $n > pqr$, we have $n/pqr > 1$, so $\Delta(n) > -1$ since $A(n) < 3$. $\square$

Proof of Theorem 1.11. Recall from Proposition 3.3 that the number of times that Δ attains -1 is given by the number of triples $(x, y, z) \in \mathbb{N}^3$ satisfying $0 \leq x \leq p-1$, $0 \leq y \leq q-1$, $0 \leq z \leq r-1$ and

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} \geq 2. \tag{3.4}$$

Our aim is to interpret this number as the number of lattice points in a tetrahedron.

Note that (3.4) is equivalent to $z \geq 2r - \frac{rx}{p} - \frac{ry}{q}$. Therefore, we seeking for the number of lattice points inside the prism $[0, p-1] \times [0, q-1] \times [0, r-1]$ that lie above the hyperplane $\Gamma :\ z = 2r - \frac{rx}{p} - \frac{ry}{q}$. A straightforward calculation shows that the hyperplane Γ intersects $x = p-1$ plane along the line $qz + ry = qr + qr/p$. Similarly, it intersects $y = q-1$ plane along the line $pz + rx = pr + pr/q$. On the $z = r-1$ plane we have the line $x + y = pq(r+1)/r$.

We depict a generic picture in Figure 2.

Now we compute the lattice points in the tetrahedron whose vertices are $A'' = (p-1, q-1, r-1)$, $B'' = (p-1, q-1, r/p + r/q)$, $C'' = (p - 1q/r + q/p, r-1)$ and $D'' = (p/r + p/q, q-1, r-1)$.

We shift the tetrahedron to the origin and simplify the coordinates:

$$\begin{aligned} A' &= (0,0,0) \\ B' &= (0, 0, \frac{pq + qr + pr - pqr}{pq}) \\ C' &= (0, \frac{pq + qr + pr - pqr}{pr}, 0) \\ D' &= (\frac{pq + qr + pr - pqr}{qr}, 0, 0) \end{aligned}$$

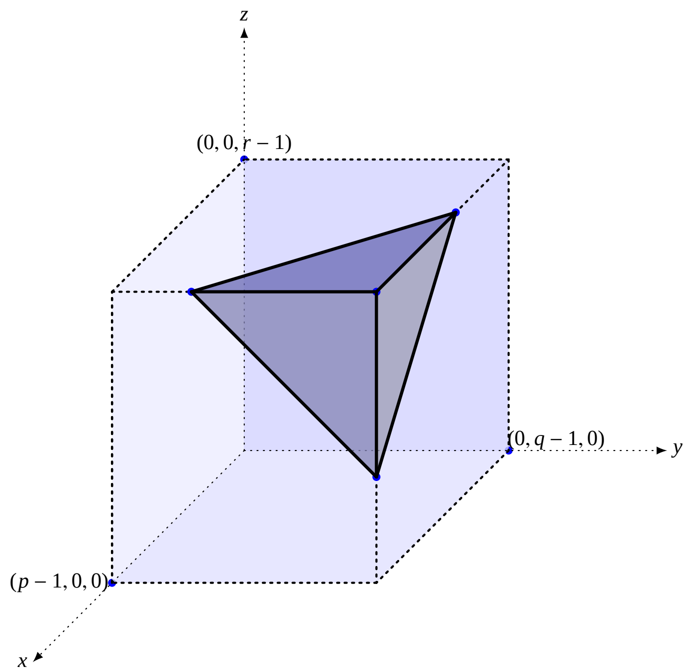

Figure 2: Tetrahedron corresponding to $\Delta = -1$

The affine transformation $x \mapsto -x, y \mapsto -y$ and $z \mapsto -z$ does not alter the number of points in the tetrahedron:

$$\begin{aligned}
A &= (0,0,0)\\
B &= (0,0,-\frac{pq+qr+pr-pqr}{pq})\\
C &= (0,-\frac{pq+qr+pr-pqr}{pr},0)\\
D &= (-\frac{pq+qr+pr-pqr}{qr},0,0)
\end{aligned}$$

Hence, the proof is complete.

□

Lemma 3.5. *For positive pairwise relatively prime integers (p,q,r) with $p < q < r$, define $N_0(p,q,r) = pqr - pq - qr - pr$. Then $N_0(p_1,q_1,r_1) \leq N_0(p_2,q_2,r_2)$, if $p_1 \leq p_2$, $q_1 \leq q_2$, and $r_1 \leq r_2$. Consequently $N_0(p,q,r) > 0$ unless $(p,q,r) = (2,3,5)$.*

Proof. Let $\mathcal{S}$ denote the set of triples of positive, pairwise relatively prime integers (p,q,r) with

$p < q < r$. Consider the partial order on $\mathcal{S}$ defined by

$$(p_1, q_1, r_1) \le (p_2, q_2, r_2) \text{ if } p_1 \le p_2,\ q_1 \le q_2,\ \text{and } r_1 \le r_2. \tag{3.6}$$

The triple $(2, 3, 5)$ is the smallest element of $(\mathcal{S}, \le)$. Define $f(x, y, z) = xyz - xy - yz - xz$ for $x \ge 2$, $y \ge 3$, $z \ge 5$, and $x \le y \le z$. Then one has $\partial f/\partial x > 0$, $\partial f/\partial y > 0$, and $\partial f/\partial z > 0$. Therefore N_0 respects the partial order $\le$ on $\mathcal{S}$. This proves the first assertion. For the second, observe that if $(p, q, r) \in \mathcal{S}$ and $(p, q, r) \ne (2, 3, 5)$, then $(p, q, r) \ge (2, 3, 7)$, or $(p, q, r) \ge (3, 4, 5)$. Since $N_0(2, 3, 7) > 0$ and $N_0(3, 4, 5) > 0$, we have $N_0(p, q, r) > 0$. □

Lemma 3.7. *Suppose* $(p, q, r) \ne (2, 3, 5)$. *Let* $N_0 = pqr - pq - pr - qr > 0$. *Then* $\Delta(N_0) = -1$, *and for all* $n > N_0$, *we have* $\Delta(n) \ge 0$.

Proof. The following congruences are easily verified:

$$\begin{aligned} p' N_0 &\equiv 1 \bmod p \\ q' N_0 &\equiv 1 \bmod q \\ r' N_0 &\equiv 1 \bmod r \end{aligned}$$

Then Lemma 3.1 implies

$$\begin{aligned} f\left(\frac{N_0 p'}{p}\right) &= \frac{p-1}{p} \\ f\left(\frac{N_0 q'}{q}\right) &= \frac{q-1}{q} \\ f\left(\frac{N_0 r'}{r}\right) &= \frac{r-1}{r}. \end{aligned}$$

Substituting these values in equation 3.2 we get

$$\Delta(N_0) = 2 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) - \left(\frac{p-1}{p} + \frac{q-1}{q} + \frac{r-1}{r}\right) = -1.$$

For the second part note that Lemma 3.1 gives the following estimate for every n

$$f\left(\frac{np'}{p}\right) + f\left(\frac{nq'}{q}\right) + f\left(\frac{nr'}{r}\right) \le \frac{p-1}{p} + \frac{q-1}{q} + \frac{r-1}{r}.$$

Suppose that $n = N_0 + s$, for $s > 0$. Then we get

$$\Delta(n) \ge 2 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) + \frac{s}{pqr} - \left(\frac{p-1}{p} + \frac{q-1}{q} + \frac{r-1}{r}\right) > -1.$$

Hence, $\Delta(n) \ge 0$ for all $n > N_0$. □

Next we prove that Δ-function gets symmetric values with respect to N_0.

Lemma 3.8. *Let $N_0 = pqr - pq - qr - pr$. For any integer i such that $0 \le i \le N_0$, we have*

$$\Delta(i) = -\Delta(N_0 - i).$$

Proof. Let a, b, and c be defined by the following conditions

$$\begin{aligned} a &\equiv ip' \bmod p, \quad 0 \le a \le p-1, \\ b &\equiv iq' \bmod q, \quad 0 \le a \le q-1, \\ c &\equiv ir' \bmod r, \quad 0 \le a \le r-1. \end{aligned}$$

Then by (3.2), we have

$$\begin{aligned} \Delta(i) &= 1 + \frac{i}{pqr} - \left(f\left(\frac{a}{p}\right) + f\left(\frac{b}{q}\right) + f\left(\frac{c}{r}\right)\right), \\ \Delta(N_0 - i) &= 2 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{i}{pqr} - \left(f\left(\frac{1-a}{p}\right) + f\left(\frac{1-b}{q}\right) + f\left(\frac{1-c}{r}\right)\right). \end{aligned}$$

After adding these two equations and plugging the definition of f in, and doing the obvious cancellations, we see that

$$\Delta(i) + \Delta(N_0 - i) = 3 - \left(\left\lceil\frac{1-a}{p}\right\rceil + \left\lceil\frac{a}{p}\right\rceil + \left\lceil\frac{1-b}{q}\right\rceil + \left\lceil\frac{b}{q}\right\rceil + \left\lceil\frac{1-c}{r}\right\rceil + \left\lceil\frac{c}{r}\right\rceil\right).$$

Hence the following lemma finishes the proof

Lemma 3.9. *Let a and p be integers such that $0 \le a \le p-1$. Then*

$$\left\lceil\frac{1-a}{p}\right\rceil + \left\lceil\frac{a}{p}\right\rceil = 1.$$

Proof. If $a \neq 0$, then the first term is 0 and the other one is 1. If $a = 0$ then the first term is 1 and the other one is 0. In both cases they add up to 1. □

□

Proof of Theorem 1.3. The first four items follow from Lemma 3.5, Lemma 3.7, Lemma 3.8, and Proposition 3.3, respectively. The proof of the second part of Lemma 3.7 shows that $\Delta(n) \ge 0$ when $(p, q, r) = (2, 3, 5)$. It remains proving the fifth item.

Let $G = G(0, pq, pr, qr)$ denote the semigroup generated by 0, pq, pr, and qr. If $n \in G$, then $n = aqr + bpr + cpq$ for some $a, b, c \geq 0$. Hence, we see that

$$\begin{aligned} np' &\equiv -a \bmod p, \\ nq' &\equiv -b \bmod q, \\ nr' &\equiv -c \bmod r. \end{aligned}$$

Let $\widetilde{a}, \widetilde{b}$ and $\widetilde{c}$ denote the residues of a, b, c modulo p, q, r, respectively. Then Lemma 3.1 implies

$$f\left(\frac{np'}{p}\right) + f\left(\frac{nq'}{q}\right) + f\left(\frac{nr'}{r}\right) = \frac{\widetilde{a}}{p} + \frac{\widetilde{b}}{q} + \frac{\widetilde{c}}{r}.$$

Plugging in (3.2), we get $\Delta(n) \geq 1$. It follows from Proposition 3.3 that $\Delta(n) = 1$, if $n \in G \cap [0, N_0]$.

Next, we must show that $G \cap [0, N_0]$ contains all the elements $n \in [0, N_0]$ with $\Delta(n) = 1$. We prove this by showing that the cardinalities of these two sets are equal. By the symmetry proven in Lemma 3.8. the number of times Δ attains $+1$ in $[0, N_0]$ is equal to the number of times Δ attains -1 in the same interval. On the other hand, we know from Theorem 1.11 that the total number of -1's of Δ is equal to the cardinality of $G \cap [0, N_0]$. Therefore, the proof is complete.

□

Next we establish the monotonicity of κ on the set of ordered triples with respect to the natural partial order $\leq$ defined in (3.6).

Proposition 3.10. *Suppose* $(p_1, q_1, r_1) \geq (p_2, q_2, r_2)$, *then* $\kappa(p_1, q_1, r_1) \geq \kappa(p_2, q_2, r_2)$.

Proof. In view of Theorem 1.11, it suffices to show that the tetrahedron T_1 corresponding to (p_1, q_1, r_1) contains the tetrahedron T_2 corresponding to (p_2, q_2, r_2). The edge of T_1 on the x-axis has length

$$l_x(T_1) = r_1\left(1 - \frac{1}{p_1} - \frac{1}{q_1}\right) - 1$$

so the assumption implies that $l_x(T_1) \geq l_x(T_2)$. By symmetry the edges on the y-, and the z-axes satisfy the same property. □

4 Generalizations

In this section we mention the necessary modifications in our results for Seifert homology spheres with four or more singular fibers. First we state the modified version of Theorem 1.3 in this case. Since its proof is identical to the case where there are three singular fibers, we omit it.

Theorem 4.1. *For $l \geq 4$, let $(p_1, p_2, \dots, p_l)$ be an l-tuple of pairwise relatively prime integers with $1 < p_1 < p_2 < \cdots < p_l$. Let $\Delta : \mathbb{N} \to \mathbb{Z}$ denote the function*

$$\Delta(n) = 1 + |e_0|n - \sum_{i=1}^{l} \left\lceil \frac{np_i'}{p_i} \right\rceil,$$

where $(e_0, p_1', p_2', \dots, p_l')$ is defined by

$$e_0 p_1 p_2 \cdots p_l + p_1' p_2 \cdots p_l + p_1 p_2' \cdots p_l + \cdots + p_1 p_2 \cdots p_l' = -1,$$

with $0 \leq p_i' \leq p_i - 1$, for all $i = 1, \dots, l$. Define the constant

$$N_0 = p_1 p_2 \cdots p_l \left((l-2) - \sum_{i=1}^{l} \frac{1}{p_i} \right).$$

Then

1. *N_0 is a positive integer.*
2. *$\Delta(n) \geq 0$, for all $n > N_0$.*
3. *$\Delta(n) = -\Delta(N_0 - n)$, for all $0 \leq n \leq N_0$.*

Next we establish the monotonicity of κ under the addition of one more singular fiber.

Proposition 4.2. *Let $(p_1, p_2, \dots, p_l, p_{l+1})$ be an $(l+1)$-tuple of pairwise relatively prime integers with $1 < p_1 < p_2 < \cdots < p_l < p_{l+1}$. Then*

$$\kappa(p_1, p_2, \dots, p_l) \leq \kappa(p_1, p_2, \dots, p_l, p_{l+1}).$$

Proof. Let Δ and Δ' be the difference terms corresponding to the l-tuple $(p_1, p_2, \dots, p_l)$ and the $(l+1)$-tuple $(p_1, p_2, \dots, p_l, p_{l+1})$ respectively. Let n be an integer with $\Delta(n) \leq 0$. We claim that $\Delta'(n) \leq \Delta(n)$. Writing the difference terms as in (3.2), we have

$$\Delta'(n) - \Delta(n) = \frac{n}{p_1 p_2 \dots p_l p_{l+1}} - \frac{n}{p_1 p_2 \dots p_l} - f\left(\frac{np_{l+1}'}{p_{l+1}}\right) \leq 0.$$

□

Our aim now is to generalize Theorem 1.11 to the case of four or more singular fibers. As in Theorem 1.11, we can realize $\kappa(p_1, p_2, \dots, p_l)$ as the count of lattice points inside a tetrahedron. The difference is that, this time we count points with appropriate multiplicities. To explain, let

$N_0(k) = p_1 \dots p_l \left(k - \sum_{i=1}^{l} \frac{1}{p_i} \right)$. For every k satisfying $N_0(k) > 0$, let T_k be the l-dimensional tetrahedron in $\mathbb{R}^l$ with one vertex at the origin and one vertex on ith axis with $x_i = \frac{N_0(k)p_i}{p_1 \dots p_l}$ for each $i = 1 \dots l$. For convenience define $T_k = \emptyset$, if $N_0(k) < 0$. Clearly, we have

$$T_0 \subset T_1 \subset T_2 \subset T_3 \cdots$$

Theorem 4.3. *Let A_j denote the number of lattice points in the set $T_j - T_{j-1}$. Then*

$$\kappa(p_1, p_2, \dots, p_l) = \sum_{j=1}^{l-2} \frac{(l-j-1)(l-j-2)}{2} A_j.$$

Proof. First we describe a convenient affine transformation of the Tetrahedra T_r. Let C be the l-dimensional cube $[0, p_1 - 1] \times [0, p_2 - 1] \cdots \times [0, p_l - 1]$. For any $r \in \{0, 1, \dots, l-1\}$, let H_r be the hyperplane defined by the equation

$$\frac{x_1}{p_1} + \frac{x_2}{p_2} + \cdots + \frac{x_l}{p_l} = l - r.$$

If $N_0(r) > 0$ then the hyperplane H_r cuts out a tetrahedron $\widetilde{T}_r$ from the cube C. The vertices of this tetrahedron are at the points

$$v_i = (p_1 - 1, p_2 - 1, \dots, p_{i-1} - 1, (1-r)p_i + \sum_{j \neq i} \frac{p_i}{p_j}, p_{i+1} - 1, \dots, p_l - 1),$$

for $i = 1, \dots, l$, and it has one more vertex at the corner point

$$v_0 = (p_1 - 1, p_2 - 1, \dots, p_l - 1).$$

For convenience define $\widetilde{T}_r = \emptyset$ if $N_0(r) < 0$.

The translation $x_i \to x_i - p_i + 1$, followed by the reflection $x_i \to -x_i$ sends each $\widetilde{T}_r$ to the tetrahedron T_r. Let $\widetilde{A}_r$ denote the number of lattice points in $\widetilde{T}_r - \widetilde{T}_{r-1}$. Then we have $A_r = \widetilde{A}_r$ for every $r \in \{1, 2, \dots, l-1\}$.

Generalizing (3.2), we write $\Delta(n)$ as a linear quasi-polynomial.

$$\Delta(n) = 1 + \frac{n}{p_1 \dots p_l} - \sum_{i=1}^{l} f\left(\frac{np_i'}{p_i}\right). \tag{4.4}$$

By Lemma 3.1, each $f\left(\frac{np_i'}{p_i}\right)$ term equals $\frac{x_i}{p_i}$ for some $x_i \in \{0, 1, \dots, p_i - 1\}$. Hence, (4.4) implies that $\Delta(n) \geq 0$ for $n \geq (l-2)p_1 \dots p_l$.

Fix $k \in \{0, 1, \ldots, l-3\}$, our aim is to find the sum of the negative values of $\Delta(n)$ for $kp_1 \ldots p_l \leq n < (k+1)p_1 \ldots p_l$. In this interval we have

$$\Delta(n) \geq -(l-k-2).$$

Fix $t \in \{1, 2, \ldots, l-k-2\}$. Let $B(k,t)$ denote the number of times $\Delta(n)$ attains the value $-t$ in the interval $kp_1 \ldots p_l \leq n < (k+1)p_1 \ldots p_l$. By (4.4), $B(k,t)$ is equal to the number solutions of the inequality

$$t + k + 2 \geq \sum_{i=1}^{l} f\left(\frac{np_i'}{p_i}\right) \geq t + k + 1, \tag{4.5}$$

with $kp_1 \ldots p_l \leq n \leq (k+1)p_1 \ldots p_l$.

By Lemma 3.1 and the Chinese remainder theorem, given any l-tuple $(x_1, \ldots, x_l)$ with $x_i \in \{0, 1, \ldots, p_i - 1\}$, there exist unique n in the interval $kp_1 \ldots p_l \leq n < (k+1)p_1 \ldots p_l$ satisfying

$$f\left(\frac{np_i'}{p_i}\right) = \frac{x_i}{p_i},$$

for every $i \in \{1, \ldots, l\}$. Hence (4.5) implies that $B(k,t)$ is equal to the number of lattice points in the cube C that lie between the hyperplanes $H_{l-t-k-2}$ and $H_{l-t-k-1}$. In other words $B(k,t) = \widetilde{A}_{l-t-k-1}$. Therefore

$$\begin{aligned}
\kappa(p_1, p_2, \ldots, p_l) = \left|\sum_{n=0}^{\infty} \min\{0, \Delta(n)\}\right| &= \left|\sum_{k=0}^{l-3} \sum_{n=kp_1\ldots p_l}^{(k+1)p_1\ldots p_l} \min\{0, \Delta(n)\}\right| \\
&= \sum_{k=0}^{l-3} \sum_{t=0}^{l-k-2} tB(k,t) = \sum_{k=0}^{l-3} \sum_{t=0}^{l-k-2} t\widetilde{A}_{l-t-k-1} \\
&= \sum_{k=0}^{l-3} \sum_{t=0}^{l-k-2} tA_{l-t-k-1} = \sum_{k=0}^{l-3} \sum_{j=1}^{l-k-2} (l-k-1-j)A_j \\
&= \sum_{j=1}^{l-2} \sum_{k=1}^{l-j-2} (l-k-1-j)A_j = \sum_{j=1}^{l-2} \sum_{k=1}^{l-j-2} (l-k-1-j)A_j \\
&= \sum_{j=1}^{l-2} \frac{(l-j-1)(l-j-2)}{2} A_j
\end{aligned}$$

□

We state the monotonicity of kappa under the natural partial order of l-tuples, generalizing Proposition 3.10.

Proposition 4.6. *Suppose* $(p_1, p_2, \ldots, p_l) \geq (q_1, q_2, \ldots, q_l)$*, then* $\kappa(p_1, p_2, \ldots, p_l)) \geq \kappa(q_1, q_2, \ldots, q_l)$.

Proof. From the discussion preceding Theorem 4.3, each tetrahedron associated to $(p_1, p_2, \ldots, p_l)$ is strictly larger than the corresponding tetrahedron associated to $(q_1, q_2, \ldots, q_l)$. Hence the monotonicity follows from the count given in Theorem 4.3. Indeed, every lattice point appearing in the calculation of $\kappa(p_1, p_2, \ldots, p_l)$ appears also in the calculation of $\kappa(q_1, q_2, \ldots, q_l)$ with a possibly bigger multiplicity. This is because of the fact that the lattice points in the smaller tetrahedra are counted with bigger multiplicity in Theorem 4.3. □

5 Topological Applications

In this section we discuss some of the topological applications of our work to the topology of 3–manifolds. Our first task is to detect the Brieskorn spheres with trivial Heegaard-Floer homology. We would like to find all Brieskorn spheres which are L–space. First we translate the condition fo being an L–space in terms of the τ function.

Proposition 5.1. *Let Y be a 3–manifold which bounds a negative definite plumbing with at most one bad vertex then Y is an L–space if and only if its tau function is increasing.*

Proof. It follows from Nemethi's work that Heegaard-Floer homology in the canonical Spin^{c}structure is given by the graded root associated root the τ function. This gives trivial homology if and only τ is increasing. Finally by Theorem 6.3 of Nemethi [8], a plumbed 3-manifold is an L–space if and only if its Heegaard-Floer homology in the canonical Spin^{c}structure is trivial. □

It is known that the 3-sphere and the Poincaré homology sphere $\Sigma(2, 3, 5)$ are examples of L-spaces. In fact it is conjectured that an irreducible integral homology sphere is an L-space if and only if it is homeomorphic to S^3 or $\Sigma(2, 3, 5)$ (with either orientation). Here we verify this conjecture for Seifert homology spheres. This was observed long before by Rustamov and independently by Eftekhary, but here we give a simpler proof.

Proof of Theorem 1.6. In view of Proposition 5.1, it suffices to prove the following: If τ function of a Seifert homology sphere $Y := \Sigma(p_1, p_2, \ldots, p_l)$ is increasing then either $l \leq 2$ (implying $Y \approx S^3$) or $l = 3$ and $(p_1, p_2, p_3) = (2, 3, 5)$. That τ is increasing is equivalent to the condition that $\Delta(n) = \tau(n + 1) - \tau(n)$ never gets a negative value. Theorem 4.1 rules out the possibility that $l \geq 4$, since $\Delta(N_0) = -\Delta(0) = -1$. If $l = 3$, Theorem 1.3 forces that $(p_1, p_2, p_3) = (2, 3, 5)$.

□

Proof of Proposition 1.10. This is an immediate consequence of Nemethi's result (Theorem 2.3), and the relationship between Heegaard-Floer homology and the Casson invariant which is shown by Ozsváth and Szabó in [11]. More precisely, they show that for every integral homology sphere Y, the Heegaard-Floer homology has such a decomposition

$$HF^+(-Y) = \mathcal{T}^+_{(d)} \oplus HF_{\mathrm{red}}(-Y),$$

where $HF_{\mathrm{red}}(-Y)$ is a finitely generated subgroup, whose Euler characteristic satisfies the following property.

$$\chi(HF_{\mathrm{red}}(-Y)) = \lambda(-Y) + \frac{d(-Y)}{2}. \tag{5.2}$$

It is shown in [12] that if Y is Seifert homology sphere (or more generally if Y bounds a negative definite plumbing with at most one bad vertex) then $HF^+(-Y)$ is supported only in even degrees. Hence $\chi(HF_{\mathrm{red}}(-Y)) = \mathrm{rank}(HF_{\mathrm{red}}(-Y))$ for every Seifert homology sphere Y. By the discussion in Section 2, we read off this quantity from the corresponding graded root directly: Simply remove the longest branch, then the number of remaining vertices is the rank of $HF_{\mathrm{red}}(-Y)$. By Theorem 2.3 the graded root is determined by the τ function. It is elementary to check that $\mathrm{rank}(\mathbb{H}_{\mathrm{red}}(R_\tau), \chi_\tau) = \min_i \tau(i) + \sum_i \max\{-\Delta(i), 0\}$ (see Corollary 3.7 of [8]). Comparing with Definition 1.7, we have $\kappa(p_1, \ldots, p_l) = \sum_i \max\{-\Delta(i), 0\}$. Substituting in (5.2), we get

$$\kappa = \lambda(-Y) + \frac{d(-Y)}{2} - \min_i \tau(i).$$

The theorem then follows from the fact that $\dfrac{d(-Y)}{2} - \min_i \tau(i)$ is the half of the degree shift term $(K^2 + s)/4$ that was discussed in Section 2. □

Proof of Theorem 1.9. Using Theorem 1.3 and Nemethi's method described in Section 2, it is easy to verify Table 1. We must show that every Seifert homology sphere has $\kappa \geq 3$, except the ones given in Table 1. Let $\Sigma(p_1, p_2, \ldots, p_l)$ be a Seifert homology sphere that does not appear in 1. Then $l \geq 3$ since only Seifert homology with less than 3 singular fibers is S^3. Suppose $l = 3$, then the triple (p_1, p_2, p_3) must be greater than or equal to one of the following triples: $(3, 5, 7)$, $(3, 4, 7)$, $(3, 5, 9)$, $(2, 7, 9)$, $(2, 5, 11)$, $(2, 5, 13)$, $(2, 5, 19)$, $(2, 3, 19)$. These triples are the immediate successors of the triples appearing in the table. It is easy to check that all of these triples have $\kappa \geq 3$, so by monotonicity we are done in the case of three singular fibers. For four and more singular fibers, we have $\kappa(p_1, p_2, \ldots, p_l) \geq \kappa(2, 3, 5, 7) \geq \kappa(3, 5, 7) \geq 4$. □

Proof of Theorem 1.8. We claim that κ is not constant on any infinite family of Seifert homology spheres each of which contains three or more singular fibers. First, consider the special case of three singular fiber families, only. Let $\{(p_n, q_n, r_n) : n = 1 \dots \infty\}$ be an infinite family of triples. Since $p_n < q_n < r_n$, the last entry r_n can not stay constant. Hence, after passing to a subsequence we may assume (p_n, q_n, r_n) is increasing and $r_n \to \infty$. This implies that $\kappa(p_n, q_n, r_n) \to \infty$ by Proposition 3.10 and its proof. In particular $\kappa(p_n, q_n, r_n)$ is not constant.

Suppose now that we have infinite family of Seifert homology spheres $(p_{1,n}, p_{2,n}, \dots, p_{l(n),n})$, with $l(n) \geq 3$ for all n. Projecting to the last three coordinates we get an infinite family of triples (p_n, q_n, r_n) with $\kappa(p_{1,n}, p_{2,n}, \dots, p_{l(n),n}) \geq \kappa(p_n, q_n, r_n)$ by Proposition 4.2. We observed that after passing to a finite subsequence $\kappa(p_n, q_n, r_n) \to \infty$. This forces that $\kappa(p_{1,n}, p_{2,n}, \dots, p_{l(n),n}) \to \infty$.

We must also show that every positive integer is realized as κ of some Seifert homology sphere. For any positive integer k, one can directly verify from Theorem 1.3 that $\kappa(2, 3, 6k + 1) = k$.

We already know that $\kappa = 0$ is realized by S^3 and the Poincaré homology sphere only. Therefore the claim about the finiteness of any family of isomorphic Heegaard-Floer homology follows from Nemethi's work explained in Section 2.

$\square$

6 Weakly Elliptic Brieskorn Spheres

In this section we use our findings to characterize all weakly elliptic Brieskorn spheres $\Sigma(p, q, r)$ in terms of their defining integers $1 < p < q < r$. We begin with introducing a new concept on numerical semigroups.

Definition 6.1. Let G be a numerical semigroup and let $n_0 \in \mathbb{N} - G$ be a positive integer. Then G is said to *alternate with respect to* n_0, if for every $x, y \in G$ such that $x < y < n_0$, there exists $z \in G$ satisfying $x < n_0 - z < y$.

Note that if G is generated by a single element a, then G alternates with respect to any $n_0 \in \mathbb{N} - G$. This notion gets more interesting if there are more than one generators. Clearly, in this case, there are only finitely many possibilities for n_0.

Lemma 6.2. *Let $G = G(a, b, c)$ be a numerical semigroup minimally generated by three relatively prime positive integers $a + 1 < b < c$, and let n_0 be a number from $\mathbb{N} - G$. Then G alternates with respect to n_0 if and only if $a < n_0 < b < c$.*

Proof. ($\Leftarrow$) Our claim is immediately proven once we replace $G(a, b, c)$ by $G(a)$.

($\Rightarrow$) Let $n_0 \in \mathbb{N} - G$ be a positive integer with respect to which G alternates. Clearly, if $n_0 < a$, then there is nothing to prove. We proceed by induction on n_0, the base case being $n_0 = a + 1$. Notice that our claim is trivially true in the base case.

Assume now that if $n_0' < n_0$ and G is alternating with respect to n_0', then $a < n_0' < b < c$. Suppose $x < y$ are from G and they are the largest elements of G that are less than n_0. Thus, there exists $z \in G$ such that $x < n_0 - z < y < n_0$. It follows that $x + z < n_0 < y + z < n_0 + z$, hence $x + z = y$. Notice that z has to be the smallest element a of G, otherwise, for $w \in G$ with $w < z$ we see that $x < w + x < y$, contradicting with the maximality of x.

We claim that G alternates with respect to $n_0' = n_0 - z$. Indeed, $n_0 - z \notin G$ and if $u < v$ are two elements from G such that $u < v < n_0 - z$, then $u + z < v + z < n_0$, hence there exists $w \in G$ such that $u + z + w < n_0 < v + z + w$. Our claim follows from this.

Now, by induction hypothesis we have that $a < n_0 - z < b < c$. But $x < n_0 - z$, so x must be a multiple of a. Then $y = x + z$ is a multiple of a. If $n_0 < b + z < y + z$, then $x < b < n_0$. Since x is the second largest element of G that is less than n_0, and since b is not a multiple of a, we obtained a contradiction. Therefore, $y + z < b + z$, or $y < b$. This implies that $n_0 < b$ and the proof is finished. $\square$

Corollary 6.3. *Let $1 < p < q < r$ be three relatively prime integers. Then the Brieskorn sphere $\Sigma(p, q, r)$ is weakly elliptic if and only if $N_0 < pr$, where $N_0 = pqr - pq - pr - qr$.*

Proof. It follows from the discussion in Section 2 that $\Sigma = \Sigma(p, q, r)$ is weakly elliptic if and only if its difference function Δ_Σ alternates along its non-zero entries in the domain $[0, N_0]$. Interpreting in terms of the numerical semigroup $G_\Sigma = G(pq, pr, qr)$ of Σ, we see that if $pq < N_0 < pr$, Δ_Σ alternates with respect to N_0 if and only if G_Σ alternates with respect to N_0. On the other hand, if $0 < N_0 < pq$, there is nothing to prove, because there only two non-zero values of Δ_Σ in $[0, N_0]$ and these are 1 and -1. $\square$

Proof of Theorem 1.13. ($\Rightarrow$) Let $\Sigma(p, q, r)$ be a weakly elliptic Brieskorn sphere. By Corollary 6.3, we know that $pqr - pq - pr - qr < pr$. Dividing by pqr, we obtain

$$1 - \frac{1}{r} - \frac{1}{q} - \frac{1}{p} < \frac{1}{q}. \tag{6.4}$$

Since $1 < p < q < r$, it follows that $1 - 3/p < 1/q$, or $1 < 1/q + 3/p$, which implies $1 < 4/p$. Thus, we conclude that $p < 4$.

We proceed with the case $p = 3$. Using (6.4) we see that $2/3 - 1/r < 2/q$. Hence, if $r \geq 6$, then $2/3 - 1/6 \leq 2/3 - 1/r < 2/q$. In other words, $1/2 < 2/q$, or $q < 4$, which is a contradiction. Therefore, $r < 6$, hence the only possibility is that $q = 4$ and $r = 5$.

Next, we look at the case when $p = 2$. Then we have

$$\frac{1}{2} - \frac{1}{r} - \frac{1}{q} < \frac{1}{q}. \tag{6.5}$$

This inequality implies that $q < 6$. There are two possibilities, $q = 3$ and $q = 5$. In the former case, we are done, already. For the latter, it follows from (6.5) that $r < 10$. Obviously, the only two possibilities are $r = 7$ and $r = 9$.

($\Leftarrow$) It follows from the definition of weakly elliptic Brieskorn spheres and Table 1 that $\Sigma(2,5,7)$, $\Sigma(2,5,9)$, $\Sigma(3,4,5)$, $\Sigma(2,3,5)$, $\Sigma(2,3,7)$, and $\Sigma(2,3,13)$ are weakly elliptic. Therefore, it is enough to show that $\Sigma(2,3,r)$, $r > 13$ is weakly elliptic.

Notice that any integer $r > 13$ that is relatively prime to 2 and 3 has the form $r = 6k \pm 1$ for some $k \geq 3$. We proceed with the case that $r = 6k + 1$. Then $N_0 = 6k - 5$. It follows that $6 < N_0 < 2(6k + 1) < 3(6k + 1)$, if $k \geq 3$. Therefore, by Corollary 6.3. $\Sigma(2,3,6k+1)$ is weakly elliptic. In the next case that $r = 6k - 1$, we have $N_0 = 6k - 7$. Similar to the previous case, $6 < N_0 < 2(6k - 1)$, if $k \geq 3$. Therefore, $\Sigma(2,3,6k-1)$ is weakly elliptic and the proof is finished. $\square$

7 Generating Function of τ

In this section we calculate the generating functions for the sequences $\tau(n)$ and $\Delta(n)$. Our main result shows that both generating functions are rational. For convenience we change our notation slightly. Let $\alpha = m/a = m_1/a_1$, $\beta = m_2/a_2$ and $\gamma = m_3/a_3$ be three rational numbers. Consider the integer valued function defined by the recurrence relation

$$\tau(n+1) = \tau(n) + 1 + |e_0|n - \left\lceil \frac{n}{\alpha} \right\rceil - \left\lceil \frac{n}{\beta} \right\rceil - \left\lceil \frac{n}{\gamma} \right\rceil \tag{7.1}$$

with $\tau(0) = 0$. In this section we prove the following theorem which determines the generating function of τ.

Theorem 7.2. *Let $(m_1, a_1), (m_2, a_2), (m_3, a_3)$ be three pairs of pairwise relatively prime positive integers, and let $\tau : \mathbb{N} \to \mathbb{Z}$ denote the τ-function defined recursively as in (7.1). Then its generating*

series $F(x) = \sum_{n\geq 1} \tau(n) x^n$ *is given by*

$$F(x) = \frac{x}{(1-x)^2} + \frac{|e_0|x^2}{(1-x)^3} - \frac{x^2}{(1-x)^2} \sum_{i=1}^{3} \frac{(1 - x^{\lfloor m_i/a_i \rfloor a_i})}{(1-x^{m_i})(1-x^{\lfloor m_i/a_i \rfloor})},$$

where $\lfloor y \rfloor$ *denotes the* floor function.

Theorem 7.2 immediately implies Theorem 1.14. The proof of Theorem 7.2 occupies the rest of the subsection. The main component of the proof is the identification of the generating function $f(x) = \sum_{n\geq 0} \left\lceil \frac{na}{m} \right\rceil x^n$ with a simple rational function. We achieve this in two steps.

Lemma 7.3. *Let* $D(x)$ *denote the polynomial* $D(x) := \sum_{i=1}^{m-1} \left\lceil \frac{ia}{m} \right\rceil x^i$. *Then*

$$f(x) = \frac{ax^m + D(x)(1-x)}{(1-x)(1-x^m)}.$$

Proof. To compute $f(x)$ in a closed form we break it into congruence classes modulo m (without worrying about convergence issues):

$$f(x) = \sum_{n\equiv 0 \mod m} \left\lceil \frac{na}{m} \right\rceil x^n + \sum_{n\equiv 1 \mod m} \left\lceil \frac{na}{m} \right\rceil x^n + \cdots + \sum_{n\equiv m-1 \mod m} \left\lceil \frac{na}{m} \right\rceil x^n,$$

or

$$f(x) = \sum_{l\geq 0} \left\lceil \frac{lma}{m} \right\rceil x^{lm} + \sum_{l\geq 0} \left\lceil \frac{(lm+1)a}{m} \right\rceil x^{lm+1} + \cdots + \sum_{l\geq 0} \left\lceil \frac{(lm+m-1)a}{m} \right\rceil x^{lm+m-1}. \tag{7.4}$$

Note that, for $i = 1, \ldots, m-1$

$$\sum_{l\geq 0} \left\lceil \frac{(lm+i)a}{m} \right\rceil x^{lm+i} = \sum_{l\geq 0} \left(la + \left\lceil \frac{ia}{m} \right\rceil \right) x^{lm+i}.$$

We separate the right hand side of (7.4) into two summations; $f(x) = A(x) + B(x)$, where

$$A(x) = \sum_{i=0}^{m-1} \sum_{l\geq 0} alx^{ml+i} \qquad \text{and} \qquad B(x) = \sum_{i=1}^{m-1} \sum_{l\geq 0} \left\lceil \frac{ia}{m} \right\rceil x^{ml+i}.$$

It is easier to find a closed formula for $A(x)$;

$$\begin{aligned} A(x) = \sum_{i=0}^{m-1} \sum_{l\geq 0} alx^{ml+i} &= \sum_{i=0}^{m-1} x^i \sum_{l\geq 0} alx^{ml} \\ &= \frac{1-x^m}{1-x} a \sum_{l\geq 0} l(x^m)^l = \frac{1-x^m}{1-x} ax^m \frac{1}{(1-x^m)^2} = \frac{ax^m}{(1-x)(1-x^m)}. \end{aligned}$$

For $B(x)$ we have

$$B(x) = \sum_{i=1}^{m-1} \sum_{l\geq 0} \left\lceil \frac{ia}{m} \right\rceil x^{ml+i} = \left(\sum_{i=1}^{m-1} \left\lceil \frac{ia}{m} \right\rceil x^i \right) \sum_{l\geq 0} x^{ml} = \left(\sum_{i=1}^{m-1} \left\lceil \frac{ia}{m} \right\rceil x^i \right) \frac{1}{1-x^m}.$$

Thus, if we define $D(x)$ as in hypothesis,

$$f(x) = A(x) + B(x) = \frac{ax^m}{(1-x)(1-x^m)} + D(x)\frac{1}{1-x^m} = \frac{ax^m + D(x)(1-x)}{(1-x)(1-x^m)}.$$

□

Lemma 7.5. *Let $m = pa + q$ with $0 \leq q < a$ then $f(x)$ can be written as*

$$f(x) = \frac{x(1-x^{pa})}{(1-x)(1-x^m)(1-x^p)}$$

Proof. From Lemma 7.3,

$$f(x) = (\sum_{i=0}^{m-1} c_i x^{i+1}) \frac{1}{(1-x)(1-x^m)},$$

where

$$c_i = \left\lceil \frac{(i+1)a}{m} \right\rceil - \left\lceil \frac{ia}{m} \right\rceil = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{p} \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$f(x) = \frac{\sum_{j=1}^{a-1} x^{jp+1}}{(1-x)(1-x^m)} = \frac{x\sum_{j=1}^{a-1} (x^p)^j}{(1-x)(1-x^m)} = \frac{x(1-x^{pa})}{(1-x)(1-x^m)(1-x^p)}.$$

□

Proof of Theorem 7.2. Let $F(x)$ and $f_i(x)$ for $i = 1, 2, 3$ denote the generating functions of $\tau(n)$ and $\left\lceil \frac{na_i}{m_i} \right\rceil$, respectively. If we multiply both sides of the equation

$$\tau(n+1) = \tau(n) + 1 + |e_0|n - \left\lceil \frac{n}{\alpha} \right\rceil - \left\lceil \frac{n}{\beta} \right\rceil - \left\lceil \frac{n}{\gamma} \right\rceil$$

by x^{n+1} and sum over $n \geq 0$, then we obtain

$$F(x) = xF(x) + \sum_{n\geq 0} x^{n+1} + \sum_{n\geq 0} |e_0|nx^{n+1} - x(f_1(x) + f_2(x) + f_3(x)).$$

Equivalently,

$$F(x) = \frac{1}{1-x}\left(\frac{x}{1-x} + \frac{|e_0|x^2}{(1-x)^2} - x(f_1(x) + f_2(x) + f_3(x)) \right).$$

Therefore, the result follows from Lemma 7.5. □

7.1 Closed form of $\tau(n)$

In this subsection we use the generating function given in Theorem 7.2 to find τ explicitly. See [9] for an alternative formula in terms of Dedekind sums.

Theorem 7.6. *The unique solution of Recurrence 7.1 is given by*

$$\begin{aligned}\tau(n) = n \quad &+ \quad |e_0|\left(\frac{n(n-1)}{2}\right) \\ &+ \quad \sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\left(-(n-\lfloor m_i/a_i\rfloor k-1)+\frac{m_i}{2}\left\lfloor\frac{n-\lfloor m_i/a_i\rfloor k-1}{m_i}\right\rfloor\right)\left(\left\lfloor\frac{n-\lfloor m_i/a_i\rfloor k-1}{m_i}\right\rfloor+1\right).\end{aligned}$$

Before starting the proof we first we state a useful lemma whose proof is left as an exercise

Lemma 7.7. *If* $g(x)=\sum_{n=0}^{\infty}c_nx^n$, *then* $\frac{x}{(1-x)^2}g(x)=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(n-j)c_j\right)x^n$.

Proof of Theorem 7.6. Let $p_i := \lfloor m_i/a_i\rfloor$ for all $i=1,2,3$. By Theorem 7.2,

$$\begin{aligned}\sum_{n=0}^{\infty}\tau(n)x^n &= \frac{x}{(1-x)^2}\left(1+\frac{|e_0|x}{1-x}-\sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\frac{x^{kp_i+1}}{1-x^{m_i}}\right)\\ &= \frac{x}{(1-x)^2}\left(1+\sum_{n=1}^{\infty}|e_0|x^n-\sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\sum_{n=0}^{\infty}x^{m_in+kp_i+1}\right)\\ &= \frac{x}{(1-x)^2}\left(\sum_{n=0}^{\infty}\xi(n)x^n-\sum_{n=0}^{\infty}\sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\epsilon_{m_i}(n-kp_i-1)x^n\right)\\ &= \frac{x}{(1-x)^2}\left(\sum_{n=0}^{\infty}c_nx^n\right),\end{aligned}$$

where $c(n)=\xi(n)-\sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\epsilon_{m_i}(n-kp_i-1)$, and

$$\epsilon_m(j)=\begin{cases}1 & \text{if } j\equiv 0 \pmod{m}\\ 0 & \text{otherwise}\end{cases} \qquad \xi(n)=\begin{cases}0 & \text{if } n=0\\ |e_0| & \text{otherwise}\end{cases}$$

By Lemma 7.7,

$$\begin{aligned}
\tau(n) &= \sum_{j=0}^{n}(n-j)c_j \\
&= \sum_{j=0}^{n}(n-j)\xi(j) - \sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\sum_{j=0}^{n}(n-j)\epsilon_{m_i}(j-kp_i-1) \\
&= n+\sum_{j=1}^{n}(n-j)|e_0| - \sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\sum_{j=0}^{\lfloor (n-kp_i-1)/m_i \rfloor} n-(m_i j+kp_i+1) \\
&= n+|e_0|n^2 - \frac{|e_0|n(n+1)}{2} - \sum_{i=1}^{3}\sum_{k=0}^{a_i-1}\left((n-kp_i-1)-\frac{m_i}{2}\left(\left\lfloor\frac{n-kp_i-1}{m_i}\right\rfloor\right)\right)\left(\left\lfloor\frac{n-kp_i-1}{m_i}\right\rfloor+1\right).
\end{aligned}$$

Hence the proof is complete. □

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